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CO2412 Computational Thinking

Normal forms
Entailment and inference
Tutorial 3.3 problem

Where opportunity creates success

Intro to logic: Propositional logic

Negation (¬)

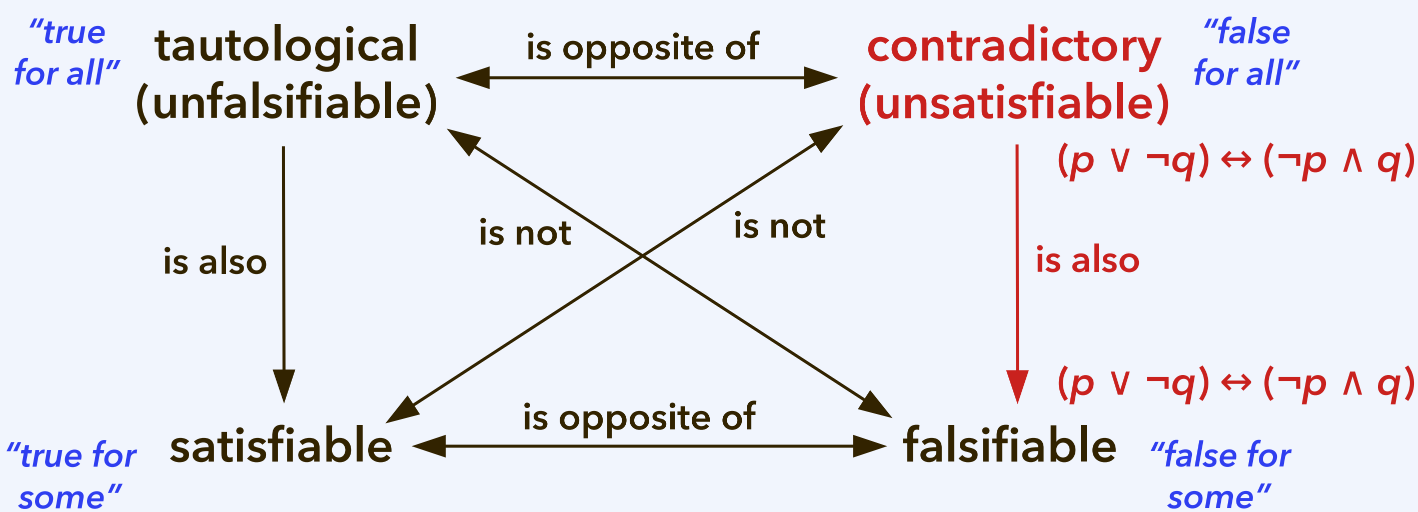
Conjunction (∧) *"and"*

Disjunction (∨) *"or"*

Implication (→)

Equivalence (↔)

p	q	$p \lor \neg q$	$\neg p \land q$	$(p \lor \neg q) \leftrightarrow (\neg p \land q)$
False	False	True	False	**False**
False	True	False	True	**False**
True	False	True	False	**False**
True	True	True	False	**False**



Intro to logic: Propositional logic

Negation ($\neg$)

Conjunction ($\wedge$) *"and"*

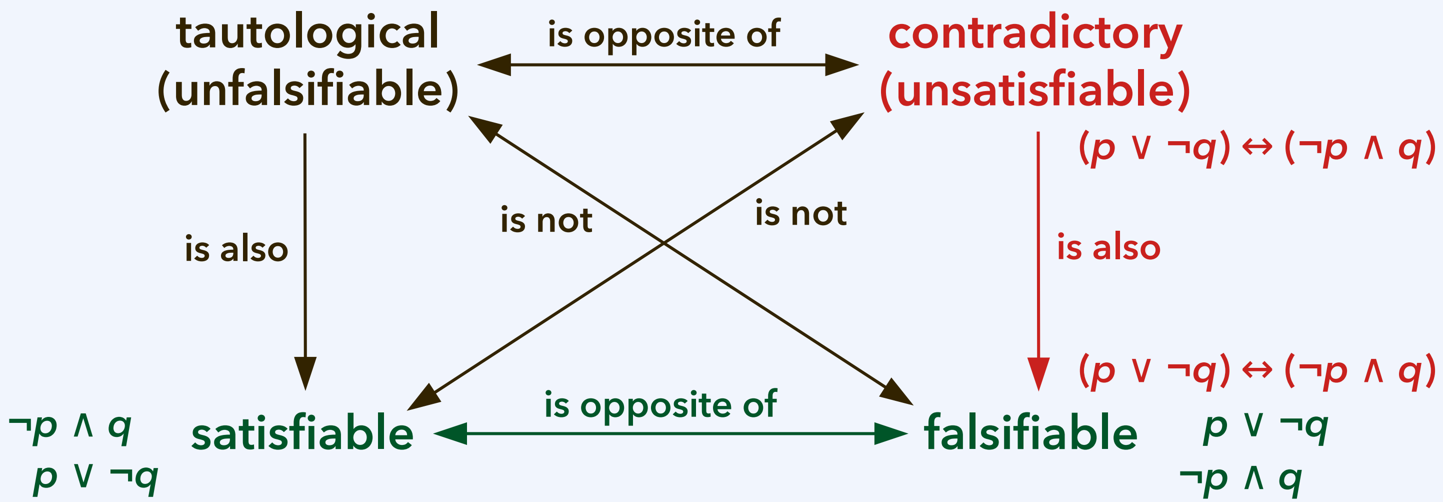
Disjunction ($\vee$) *"or"*

Implication ($\rightarrow$)

Equivalence ($\leftrightarrow$)

p	q	$p \vee \neg q$	$\neg p \wedge q$
False	False	**True**	**False**
False	True	**False**	**True**
True	False	**True**	**False**
True	True	**True**	**False**

Statements that are both satisfiable and falsifiable are called **contingent**.



Intro to logic: Propositional logic

Negation (¬)

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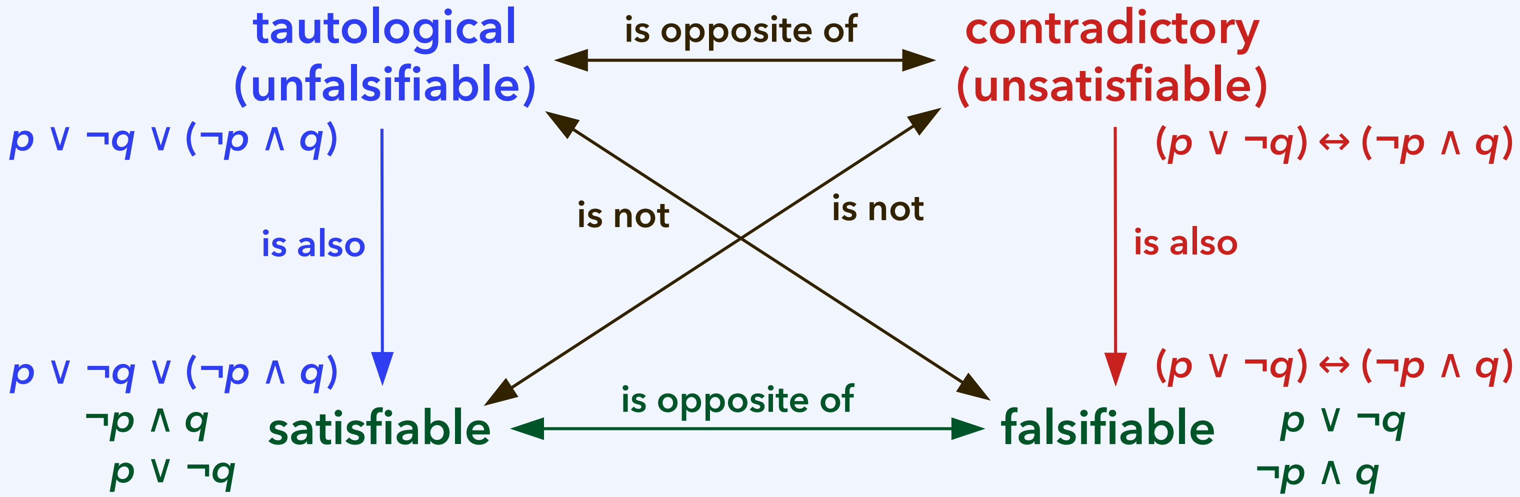
Equivalence (↔)

p	q	$p \vee \neg q$	$\neg p \wedge q$	$(p \vee \neg q) \vee (\neg p \wedge q)$
False	False	True	False	**True**
False	True	False	True	**True**
True	False	True	False	**True**
True	True	True	False	**True**



Normal forms in propositional logic

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Syntactic equivalence rules

Every statement has its unique, well-defined semantics: Its truth table.

But the obverse is not true. There are many formulas for the same truth table.
Statements with the same truth table are called **semantically equivalent**.

Examples:

- $\neg(p \lor (\neg p \land \neg q)) \equiv \neg p \land \neg(\neg p \land \neg q) \equiv \neg p \land (\neg\neg p \lor \neg\neg q) \equiv \neg p \land (p \lor q)$
- $\neg p \land (p \lor q) \equiv (\neg p \land p) \lor (\neg p \land q) \equiv \text{False} \lor (\neg p \land q) \equiv \neg p \land q$

Associative laws

$R \lor (S \lor T) \equiv (R \lor S) \lor T,$
$R \land (S \land T) \equiv (R \land S) \land T,$

Distributive laws

$R \land (S \lor T) \equiv (R \land S) \lor (R \land T),$
$R \lor (S \land T) \equiv (R \lor S) \land (R \lor T),$

De Morgan's laws

$\neg(R \lor S) \equiv \neg R \land \neg S$
$\neg(R \land S) \equiv \neg R \lor \neg S$

here we usually omit the parentheses right away

Syntactic equivalence rules

Every statement has its unique, well-defined semantics: Its truth table.

But the obverse is not true. There are many formulas for the same truth table. **Statements with the same truth table** are called **semantically equivalent**.

Notation: $R \equiv S$ ("R and S are semantically equivalent")

Syntactic equivalence rules (*i.e.*, rules based on the statement structure) include:

Obvious equivalences $\neg\neg S \equiv S \vee S \equiv S \wedge S \equiv S$, $R \wedge S \equiv S \wedge R$, $S \vee \neg S \equiv S \vee \text{True} \equiv \text{True}$, $S \wedge \neg S \equiv S \wedge \text{False} \equiv \text{False}$, $\neg\text{True} \equiv \text{False}$

Associative laws

$R \vee (S \vee T) \equiv (R \vee S) \vee T$,

$R \wedge (S \wedge T) \equiv (R \wedge S) \wedge T$,

Distributive laws

$R \wedge (S \vee T) \equiv (R \wedge S) \vee (R \wedge T)$,

$R \vee (S \wedge T) \equiv (R \vee S) \wedge (R \vee T)$,

De Morgan's laws

$\neg(R \vee S) \equiv \neg R \wedge \neg S$

$\neg(R \wedge S) \equiv \neg R \vee \neg S$

Definition of implication and equivalence

$R \rightarrow S \equiv \neg R \vee S$, $R \leftrightarrow S \equiv (R \rightarrow S) \wedge (S \rightarrow R)$

Syntactic equivalence rules: Example

"(p or q) and (not-p or q)"

Attempt to simplify the statement $(p \lor q) \land (\neg p \lor q)$:

$$(p \lor q) \land (\neg p \lor q) \equiv ((p \lor q) \land \neg p) \lor ((p \lor q) \land q)$$

$$\equiv (p \land \neg p) \lor (q \land \neg p) \lor (p \land q) \lor (q \land q)$$

$$\equiv \text{False} \lor (q \land (\neg p \lor p)) \lor q$$

Obvious equivalences $\neg\neg S \equiv S \lor S \equiv S \land S \equiv S$, $R \land S \equiv S \land R$, $S \lor \neg S \equiv S \lor \text{True} \equiv \text{True}$, $S \land \neg S \equiv S \land \text{False} \equiv \text{False}$, $\neg\text{True} \equiv \text{False}$

Associative laws

$R \lor (S \lor T) \equiv (R \lor S) \lor T$,

$R \land (S \land T) \equiv (R \land S) \land T$,

Distributive laws

$R \land (S \lor T) \equiv (R \land S) \lor (R \land T)$,

$R \lor (S \land T) \equiv (R \lor S) \land (R \lor T)$,

De Morgan's laws

$\neg(R \lor S) \equiv \neg R \land \neg S$

$\neg(R \land S) \equiv \neg R \lor \neg S$

Definition of implication and equivalence

$R \rightarrow S \equiv \neg R \lor S$, $R \leftrightarrow S \equiv (R \rightarrow S) \land (S \rightarrow R)$

Syntactic equivalence rules: Example

"(p or q) and (not-p or q)"

Attempt to simplify the statement $(p \lor q) \land (\neg p \lor q)$:

$$
\begin{aligned}
(p \lor q) \land (\neg p \lor q) &\equiv ((p \lor q) \land \neg p) \lor ((p \lor q) \land q) \\
&\equiv (p \land \neg p) \lor (q \land \neg p) \lor (p \land q) \lor (q \land q) \\
&\equiv \text{False} \lor (q \land (\neg p \lor p)) \lor q \\
&\equiv (q \land \text{True}) \lor q \\
&\equiv q \lor q \equiv q
\end{aligned}
$$

This is tedious. It would have been simpler to evaluate the truth table.

p	q	$p \lor q$	$\neg p \lor q$	$(p \lor q) \land (\neg p \lor q)$
False	False	False	True	False
False	True	True	True	True
True	False	True	False	False
True	True	True	True	True

Semantic equivalence

Every statement has its unique, well-defined semantics: Its truth table.

But the obverse is not true. There are many formulas for the same truth table. **Statements with the same truth table** are called **semantically equivalent**.

p	*q*	**shared semantics (meaning) of these statements:**
False	False	True
False	True	True
True	False	False
True	True	True

"not-p or (not-q implies q)"

$\neg p \lor (\neg q \to q)$

"not (not-q and not-not-p)"

$\neg(\neg q \land \neg\neg p)$

"(p implies q) and (q or not-q)"

$(p \to q) \land (q \lor \neg q)$

Semantic equivalence

Attention to pitfall: "Semantic equivalence" (of statements R and S) is not the same as the "logical equivalence" operator (as *part of* a statement).

Connection between the two: $R \equiv S$ if and only if statement $R \leftrightarrow S$ is a tautology.

p	***q***	***r***
False	False	False
False	False	True
False	True	False
False	True	True
True	False	False
True	False	True
True	True	False
True	True	True

Notation:

$R \equiv S$ ("R and S are semantically equivalent")

How can we check whether two statements are semantically equivalent?

… compute their truth tables.

However, that requires **exponential time** as a function of the number of atomic statements n:

There are 2^n possible **valuations**, *i.e.*, truth value assignments to the atomic statements. Therefore the truth table always has 2^n entries.

Normal forms: From truth tables to statements

Assume that a truth table is given.

How can we easily construct a logical statement with the given semantics?

p	q	r	**specified semantics**	
False	False	False	True	← made true by $\neg p \wedge \neg q \wedge \neg r$
False	False	True	False	
False	True	False	False	
False	True	True	True	← made true by $\neg p \wedge q \wedge r$
True	False	False	True	← made true by $p \wedge \neg q \wedge \neg r$
True	False	True	False	
True	True	False	True	← made true by $p \wedge q \wedge \neg r$
True	True	True	False	

Normal forms: From truth tables to statements

$$(\neg p \land \neg q \land \neg r) \lor (\neg p \land q \land r) \lor (p \land \neg q \land \neg r) \lor (p \land q \land \neg r)$$

The statement above is in **full disjunctive normal form (full DNF)**.
Every truth table has exactly one unique representation in full DNF.

p	*q*	*r*	specified semantics	
False	False	False	True	← made true by $\neg p \land \neg q \land \neg r$
False	False	True	False	
False	True	False	False	
False	True	True	True	← made true by $\neg p \land q \land r$
True	False	False	True	← made true by $p \land \neg r$
True	False	True	False	
True	True	False	True	← made true by $p \land \neg r$
True	True	True	False	

shorter version, in DNF, but not full DNF:

$$(\neg p \land \neg q \land \neg r) \lor (\neg p \land q \land r) \lor (p \land \neg r)$$

Disjunctive normal form (DNF)

conjunctive clause, conjunctive clause, conjunctive clause, conjunctive clause

$$(\neg p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r)$$

The statement above is in **full disjunctive normal form (full DNF)**.
Each "True" entry in a truth table produces a conjunctive clause in full DNF.

p	***q***	***r***	
False	False	False	True
False	False	True	False
False	True	False	False
False	True	True	True
True	False	False	True
True	False	True	False
True	True	False	True
True	True	True	False

Atomic statements and their negations, such as p, $\neg p$, q, $\neg q$, r, $\neg r$, are called **literals**. A conjunction of literals is called a **conjunctive clause**.

A statement is in **DNF** if it is a disjunction of conjunctive clauses. It is in **full DNF** if all conjunctive clauses contain all atomic statements.

Disjunctive normal form (DNF)

The example below has six True entries in the truth table.
It corresponds to a full DNF form with six clauses:

$$(\neg p \land \neg q \land \neg r) \lor (\neg p \land \neg q \land r) \lor (\neg p \land q \land r) \lor (p \land \neg q \land \neg r) \lor (p \land q \land \neg r) \lor (p \land q \land r)$$

p	***q***	***r***	**specified semantics**
False	False	False	True
False	False	True	True
False	True	False	False
False	True	True	True
True	False	False	True
True	False	True	False
True	True	False	True
True	True	True	True

Observation:

With n atomic statements, the truth table has 2^n entries.

If almost all entries are "True," it would be far more efficient to construct a standard formula based on the few entries that are "False."

Conjunctive normal form (CNF)

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Observation:

- The statement $p \lor \neg q \lor r$ makes the third row False, as desired;
- The statement $\neg p \lor q \lor \neg r$ makes the sixth row False, as desired;
- These are both **disjunctions of literals**, that is, **disjunctive clauses**.

p	***q***	***r***	**specification**	$p \lor \neg q \lor r$ *"p or not-q or r"*	$\neg p \lor q \lor \neg r$ *"not-p or q or not-r"*
False	False	False	True	True	True
False	False	True	True	True	True
False	True	False	False ←	False	True
False	True	True	True	True	True
True	False	False	True	True	True
True	False	True	False ←	True	False
True	True	False	True	True	True
True	True	True	True	True	True

Conjunctive normal form (CNF)

disjunctive clause — disjunctive clause

$(p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg r)$

The statement above is in **full conjunctive normal form (full CNF)**.

Each "False" entry in a truth table produces one disjunctive clause in full CNF.

p	q	r	specification	$(p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg r)$ *"(p or not-q or r) and (not-p or q or not-r)"*
False	False	False	True	True
False	False	True	True	True
False	True	False	False ←	False
False	True	True	True	True
True	False	False	True	True
True	False	True	False ←	False
True	True	False	True	True
True	True	True	True	True

Example: Transformation into normal forms

original formula	full disjunctive normal form	full CNF
$\neg p \lor (\neg q \to q)$	$(\neg p \land \neg q) \lor (\neg p \land q) \lor (p \land q)$	$(\neg p \lor q)$
"not-p or (not-q implies q)"	*"(not-p and not-q) or (not-p and q) or (p and q)"*	*"not-p or q"*

p	q	$\neg q \to q$	$\neg p \lor (\neg q \to q)$	
False	False	False	True	← is True for conjunctive clause $\neg p \land \neg q$
False	True	True	True	← is True for conjunctive clause $\neg p \land q$
True	False	False	False	← is False for disjunctive clause $\neg p \lor q$
True	True	True	True	← is True for conjunctive clause $p \land q$

Normal forms: Summary

Valuation: Assignment of truth values to all the atomic statements. For n atomic statements, there are 2^n possible valuations. Each row of a truth table corresponds to one valuation.

Literals: Atomic statements p, q, …, and their negations $\neg p$, $\neg q$, …

Clauses: A conjunction ("and") of literals, such as $p \wedge \neg q \wedge \neg r$, is a conjunctive clause. A disjunction ("or") of literals, such as $\neg p \vee q \vee r$, is a disjunctive clause.

DNF:

- A statement is in DNF if it is a **disjunction of conjunctive clauses**.
- It is in **full DNF** if all atomic statements appear in all conjunctive clauses.
- The full DNF version of a truth table has one clause per **True** valuation.

CNF:

- A statement is in CNF if it is a **conjunction of disjunctive clauses**.
- It is in **full CNF** if all atomic statements appear in all disjunctive clauses.

Normal forms: Tautologies and contradictions

CNF:

- A statement is in CNF if it is a **conjunction of disjunctive clauses**.
- It is in **full CNF** if all atomic statements appear in all disjunctive clauses.
- The full CNF version of a truth table has one clause per **False** valuation.
- For a **tautology**, no False valuations exist; the full CNF is just "True."

- Tautology in full CNF: True
- Contradiction in full CNF: $(\neg p \lor \neg q) \land (\neg p \lor q) \land (p \lor \neg q) \land (p \lor q)$

DNF:

- A statement is in DNF if it is a **disjunction of conjunctive clauses**.
- It is in **full DNF** if all atomic statements appear in all conjunctive clauses.
- The full DNF version of a truth table has one clause per **True** valuation.
- For a **contradiction**, no True valuations exist; the full DNF is just "False."

- Tautology in full DNF: $(\neg p \land \neg q) \lor (\neg p \land q) \lor (p \land \neg q) \lor (p \land q)$
- Contradiction in full DNF: False

Entailment and inference

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Entailment

The statement R **entails** the statement S if *all valuations that fulfill R also fulfill S*. In other words: If R is True, we know that S is True as well.

A **valuation** for which a statement S becomes True is also called a **model** of S.

Semantic equivalence	**Entailment**
$R \equiv S$	$R \vDash S$
("The statements R and S are semantically equivalent")	("The premise R entails the conclusion S")
***R* and *S* have the same models:** Their truth tables are the same.	**All models of *R* are also models of *S*.** S may still be True where R is False, *i.e.*, S may have more models than R.
The statement $R \leftrightarrow S$ is a tautology.	The statement $R \rightarrow S$ is a tautology.

Entailment

It is also possible for multiple statements (premises) to **entail** another statement (conclusion). That occurs if *all valuations that fulfill all the premises also fulfill the conclusion*. If the premises are *all* True, the conclusion is True as well.

Remark

If S is a **tautology**, it must always be True; for this, no premises are needed.

We can then use the same notation, but without premises:

$$\models S$$

Entailment

$$R_0, R_1, \ldots \models S$$

("The premises R_0, R_1, … entail the conclusion S")

All models of all premises are also models of S.

Where R_0, R_1 etc. are all True, S must also be True.

But S can have more True entries than $R_0 \land R_1 \land \ldots$

The statement $(R_0 \land R_1 \land \ldots) \rightarrow S$ is a tautology.

Entailment: Example

The premises "p implies q" and "not-q" together entail the consequence "not-p".

1) Show that $p \rightarrow q,\ \neg q \vDash \neg p$.

p	q	$(p \rightarrow q)$	$\neg q$	$\neg p$
False	False	**True**	**True**	**True**
False	True	**True**	False	True
True	False	False	**True**	False
True	True	**True**	False	False

The premise "(p implies q) and not-q" entails the consequence "not-p".

2) Show that $((p \rightarrow q) \wedge \neg q) \vDash \neg p$.

p	q	$(p \rightarrow q)$	$\neg q$	$(p \rightarrow q) \wedge \neg q$	$\neg p$
False	False	True	True	**True**	**True**
False	True	True	False	False	True
True	False	False	True	False	False
True	True	True	False	False	False

Entailment: Example

The statement "((p implies q) and not-q) implies not-p" is a tautology.

3) Show that $\models ((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$.

p	q	$(p \rightarrow q) \wedge \neg q$	$\neg p$	$((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$
False	False	True	True	**True**
False	True	False	True	**True**
True	False	False	False	**True**
True	True	False	False	**True**

The premise "(p implies q) and not-q" entails the consequence "not-p".

2) Show that $((p \rightarrow q) \wedge \neg q) \models \neg p$.

p	q	$(p \rightarrow q)$	$\neg q$	$(p \rightarrow q) \wedge \neg q$	$\neg p$
False	False	True	True	**True**	**True**
False	True	True	False	False	True
True	False	False	True	False	False
True	True	True	False	False	False

Inference rules for propositional logic

The pattern that we have found for semantic equivalence repeats itself for entailment: We can determine entailment *semantically*, from the truth tables.

We can also determine it *syntactically*, using **inference rules** such as:

$$R \vDash \text{True}$$

$$\text{False} \vDash S$$

$$R \to S,\ R \vDash S$$

$$S \to R,\ \neg R \vDash \neg S$$

$$R,\ S \vDash R \land S$$

$$R \lor S,\ \neg R \lor T \vDash S \lor T$$

$$R \land S \vDash S$$

$$R \vDash R \lor S$$

$$R \vDash \neg R \to S$$

$$R \vDash S \to R$$

All the rules for semantic equivalence can be listed here again, in both ways, such as:

$$S \vDash \neg\neg S \qquad \neg\neg S \vDash S$$

The process of finding statements that are entailed by a given set of premises (also called axioms in some contexts) is called **inference**.

BFS implementation (tutorial 3.3 problem)

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See "breadth-first-search" Jupyter Notebook

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