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CO2412
Computational Thinking

Logical operators & truth tables: Repetition
Syntactic transformations: Repetition
Entailment and normal forms: Repetition

Where opportunity creates success

Logical operators & truth tables: Repetition

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8th February 2022

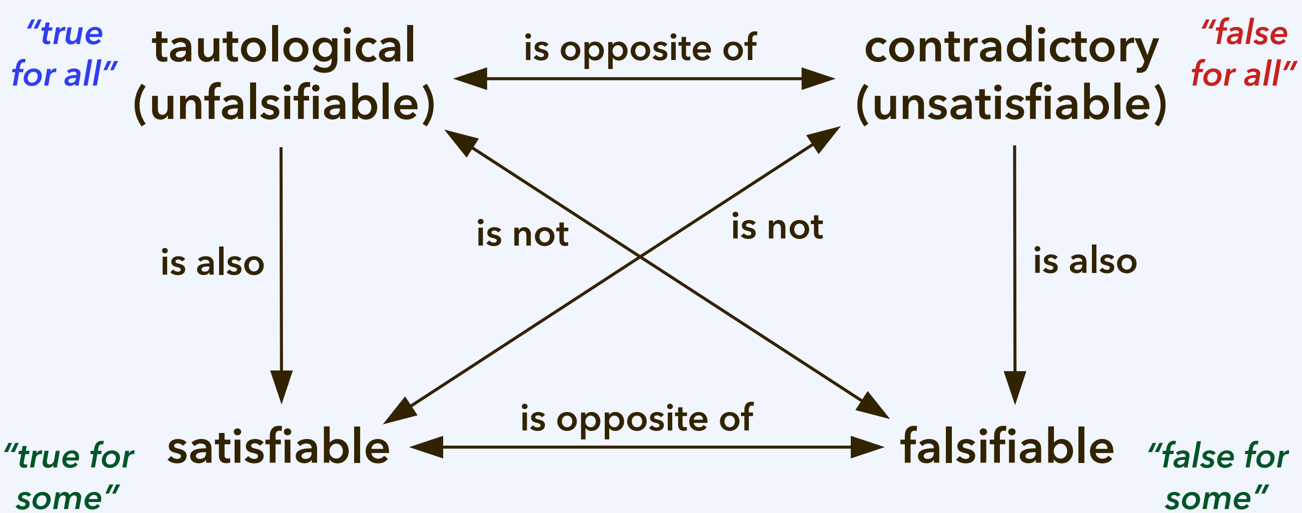
Operators and truth tables: Summary

Negation (¬)

Conjunction (∧)
Disjunction (∨)

Implication (→)
Equivalence (↔)

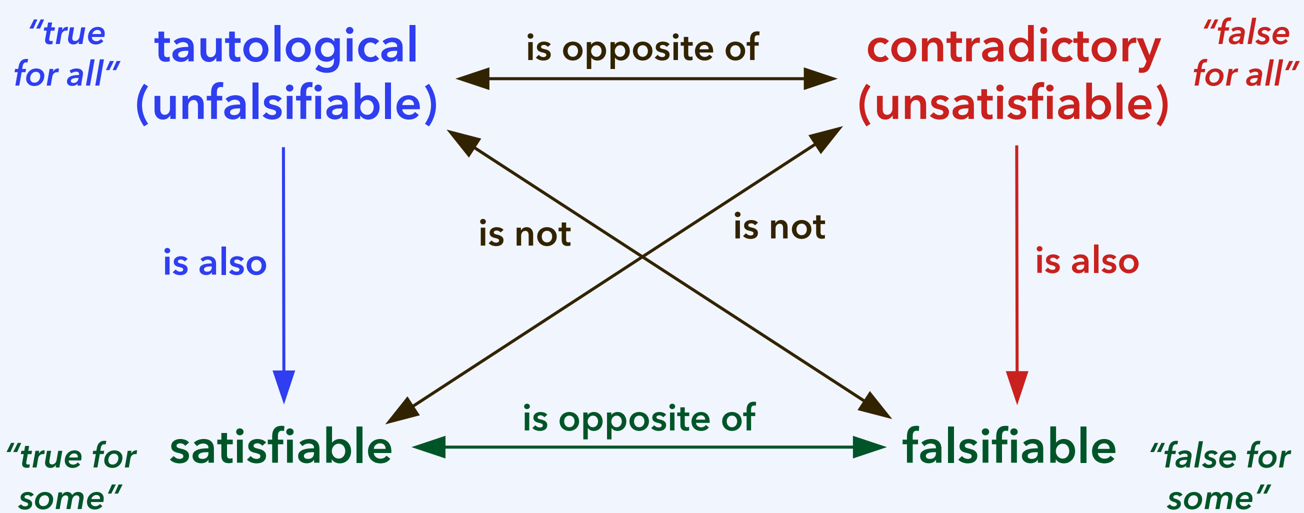
p	q	$p \vee q$	$p \wedge q$	$p \rightarrow q$	$p \leftrightarrow q$
False	False	False	False	True	True
False	True	True	False	True	False
True	False	True	False	False	False
True	True	True	True	True	True



Operators and truth tables: Summary

The negation of a **satisfiable** statement is **falsifiable**, and vice versa.
The negation of a **tautology** is a **contradiction**, and vice versa.

p	q	$p \land q$	$\neg(p \land q)$	$p \land q \land \neg(p \land q)$	$\neg(p \land q \land \neg(p \land q))$
False	False	False	True	False	True
False	True	False	True	False	True
True	False	False	True	False	True
True	True	True	False	False	True



Operators and truth tables: Examples

Examples 1 & 2: Are the statements satisfiable/falsifiable? Provide truth tables.

1) $S_1 = (\neg p \lor \neg q) \land (\neg p \lor q) \land (p \lor q)$.

p	q	$\neg p \lor \neg q$	$\neg p \lor q$	$p \lor q$
False	False	True	True	**False**
False	True	True	True	True
True	False	True	**False**	True
True	True	**False**	True	True

Observation: This statement is in **full CNF**. Each clause makes one entry False.

2) $S_2 = \neg(\neg(p \to q) \to p)$.

p	q	$p \to q$
False	False	True
False	True	True
True	False	False
True	True	True

Operators and truth tables: Examples

Examples 1 & 2: Are the statements satisfiable/falsifiable? Provide truth tables.

1) $S_1 = (\neg p \vee \neg q) \wedge (\neg p \vee q) \wedge (p \vee q)$. Answer: S_1 is **satisfiable** and **falsifiable**.

p	q	$\neg p \vee \neg q$	$\neg p \vee q$	$p \vee q$	S_1
False	False	True	True	**False**	False
False	True	True	True	True	**True**
True	False	True	**False**	True	False
True	True	**False**	True	True	False

We also say: S_1 is **contingent**, *i.e.*, it is not tautological and not contradictory.

2) $S_2 = \neg(\neg(p \rightarrow q) \rightarrow p)$.

p	q	$p \rightarrow q$
False	False	**True**
False	True	**True**
True	False	False
True	**True**	**True**

Observation: We evaluate **logical implication (→)** as follows. If the left hand side becomes False, it is True. If the right hand side becomes True, it is also True. Otherwise it is False.

Operators and truth tables: Examples

Examples 1 & 2: Are the statements satisfiable/falsifiable? Provide truth tables.

1) $S_1 = (\neg p \lor \neg q) \land (\neg p \lor q) \land (p \lor q)$. Answer: S_1 is **satisfiable** and **falsifiable**.

p	q	$\neg p \lor \neg q$	$\neg p \lor q$	$p \lor q$	S_1
False	False	True	True	**False**	False
False	True	True	True	True	**True**
True	False	True	**False**	True	False
True	True	**False**	True	True	False

We also say: S_1 is **contingent**, *i.e.*, it is not tautological and not contradictory.

2) $S_2 = \neg(\neg(p \rightarrow q) \rightarrow p)$. S_2 is **unsatisfiable** (**contradictory**); it is **falsifiable**.

p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg(p \rightarrow q) \rightarrow p$	S_2
False	False	True	False	True	False
False	True	True	False	True	False
True	False	False	True	True	False
True	True	True	False	True	False

Operators and truth tables: Discussion

How about the following statement?

$$p \lor q \lor (r \rightarrow s) \lor \neg q$$

Look at the statement and classify it. Is it satisfiable? Is it falsifiable?

What sort of valuation might satisfy the statement (make it True)?
What sort of valuation might falsify the statement (make it False)?

Don't try to build the whole truth table as there are $2^4 = 16$ possible valuations for the four atomic statements p, q, r, and s that occur in the statement.

Operators and truth tables: Examples

Examples 3: Are the statements satisfiable/falsifiable?
Answer without computing the whole truth table, and justify the answer.

a) $S_{3a} = p \lor (q \land ((r \leftrightarrow s) \rightarrow (s \rightarrow \neg p)) \land (q \leftrightarrow s))$.

b) $S_{3b} = (p \leftrightarrow \neg q) \land (q \leftrightarrow \neg r) \land (r \leftrightarrow \neg p)$.

c) $S_{3c} = p_0 \rightarrow (p_1 \rightarrow (p_2 \rightarrow (p_3 \rightarrow (p_4 \rightarrow (p_5 \rightarrow (p_6 \rightarrow (p_7 \rightarrow p_0)))))))$.

Operators and truth tables: Examples

Examples 3: Are the statements satisfiable/falsifiable?
Answer without computing the whole truth table, and justify the answer.

a) $S_{3a} = p \lor (q \land ((r \leftrightarrow s) \rightarrow (s \rightarrow \neg p)) \land (q \leftrightarrow s))$.

Statement S_{3a} is True if p is True. It is False if p and q are both False.

Statement S_{3a} is therefore contingent (both satisfiable and falsifiable).

b) $S_{3b} = (p \leftrightarrow \neg q) \land (q \leftrightarrow \neg r) \land (r \leftrightarrow \neg p)$.

For statement S_{3b} to be True, p must have the same truth value as $\neg q$, which must be that of r, which must be that of $\neg p$. That is impossible. S_{3b} is therefore unsatisfiable.

c) $S_{3c} = p_0 \rightarrow (p_1 \rightarrow (p_2 \rightarrow (p_3 \rightarrow (p_4 \rightarrow (p_5 \rightarrow (p_6 \rightarrow (p_7 \rightarrow p_0)))))))$.

It is easy to satisfy S_{3c}: Set p_0 to False, and S_{3c} becomes True.

For S_{3c} to become False, working inward, we find that p_0, p_1, p_2, …, p_7 must be True, while p_0 must be False. Since p_0 cannot be both True and False, S_{3c} is tautological.

Syntactic transformations: Repetition

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Syntactic transformations: Summary

Every statement has its unique, well-defined semantics: Its truth table.

But the obverse is not true. There are many formulas for the same truth table. Statements with the **same truth table** are called **semantically equivalent**.

Notation: $R \equiv S$ ("R and S are semantically equivalent")

Syntactic equivalence rules (*i.e.*, rules based on the statement structure) include:

Obvious equivalences $S \vee S \equiv S \wedge S \equiv \neg\neg S \equiv S$, $R \wedge S \equiv S \wedge R$,
$S \vee \neg S \equiv S \vee \mathrm{True} \equiv \mathrm{True}$, $S \wedge \neg S \equiv S \wedge \mathrm{False} \equiv \mathrm{False}$, $\neg\mathrm{True} \equiv \mathrm{False}$

Associative laws

$R \vee (S \vee T) \equiv (R \vee S) \vee T$,
$R \wedge (S \wedge T) \equiv (R \wedge S) \wedge T$,

Distributive laws

$R \wedge (S \vee T) \equiv (R \wedge S) \vee (R \wedge T)$,
$R \vee (S \wedge T) \equiv (R \vee S) \wedge (R \vee T)$,

De Morgan's laws

$\neg(R \vee S) \equiv \neg R \wedge \neg S$
$\neg(R \wedge S) \equiv \neg R \vee \neg S$

Definition of implication and equivalence

$R \rightarrow S \equiv \neg R \vee S \equiv \neg(R \wedge \neg S)$, $R \leftrightarrow S \equiv (R \rightarrow S) \wedge (S \rightarrow R) \equiv (R \wedge S) \vee (\neg R \wedge \neg S)$

Syntactic transformations: Summary

Literals: Atomic statements p, q, …, and their negations $\neg p$, $\neg q$, …

De Morgan's laws can be used to push negations further inside a composite statement, eventually producing a form where negations occur in literals only.

$$\neg(S_1 \vee S_2 \vee S_3 \vee \ldots) \equiv \neg S_1 \wedge \neg S_2 \wedge \neg S_3 \wedge \ldots$$
$$\neg(S_1 \wedge S_2 \wedge S_3 \wedge \ldots) \equiv \neg S_1 \vee \neg S_2 \vee \neg S_3 \vee \ldots$$

Obvious equivalences $S \vee S \equiv S \wedge S \equiv \neg\neg S \equiv S$, $R \wedge S \equiv S \wedge R$, $S \vee \neg S \equiv S \vee \text{True} \equiv \text{True}$, $S \wedge \neg S \equiv S \wedge \text{False} \equiv \text{False}$, $\neg\text{True} \equiv \text{False}$

Associative laws

$R \vee (S \vee T) \equiv (R \vee S) \vee T$,
$R \wedge (S \wedge T) \equiv (R \wedge S) \wedge T$,

Distributive laws

$R \wedge (S \vee T) \equiv (R \wedge S) \vee (R \wedge T)$,
$R \vee (S \wedge T) \equiv (R \vee S) \wedge (R \vee T)$,

De Morgan's laws

$\neg(R \vee S) \equiv \neg R \wedge \neg S$
$\neg(R \wedge S) \equiv \neg R \vee \neg S$

Definition of implication and equivalence

$R \to S \equiv \neg R \vee S \equiv \neg(R \wedge \neg S)$, $R \leftrightarrow S \equiv (R \to S) \wedge (S \to R) \equiv (R \wedge S) \vee (\neg R \wedge \neg S)$

Syntactic transformations: Examples

Example 4: a) Transform the statement $S_4 = (p \land q) \rightarrow (q \leftrightarrow r)$ into a semantically equivalent form where only the negation, disjunction, and conjunction operators are used; negations should only occur within literals.

b) Simplify the statement as far as possible. **c)** Transform $\neg S_4$ in the same way.

$$\begin{aligned} S_4 &\equiv \neg(p \land q) \lor (q \leftrightarrow r) \\ &\equiv \neg p \lor \neg q \lor (q \land r) \lor (\neg q \land \neg r) \end{aligned}$$

Observation: This is a statement in DNF.
Note that it is not in full DNF.

De Morgan's laws

$\neg(R \lor S) \equiv \neg R \land \neg S$

$\neg(R \land S) \equiv \neg R \lor \neg S$

Definition of implication and equivalence

$R \rightarrow S \equiv \neg R \lor S \equiv \neg(R \land \neg S)$, $R \leftrightarrow S \equiv (R \rightarrow S) \land (S \rightarrow R) \equiv (R \land S) \lor (\neg R \land \neg S)$

Syntactic transformations: Examples

Example 4: a) Transform the statement $S_4 = (p \wedge q) \rightarrow (q \leftrightarrow r)$ into a semantically equivalent form where only the negation, disjunction, and conjunction operators are used; negations should only occur within literals.

b) Simplify the statement as far as possible. **c)** Transform $\neg S_4$ in the same way.

$$
\begin{aligned}
S_4 &\equiv \neg(p \wedge q) \vee (q \leftrightarrow r) \\
&\equiv \neg p \vee \neg q \vee (q \wedge r) \vee (\neg q \wedge \neg r) \\
&\equiv \neg p \vee ((\neg q \vee q) \wedge (\neg q \vee r)) \vee (\neg q \wedge \neg r) \\
&\equiv \neg p \vee (\mathrm{True} \wedge (\neg q \vee r)) \vee (\neg q \wedge \neg r) \\
&\equiv \neg p \vee \neg q \vee r \vee (\neg q \wedge \neg r)
\end{aligned}
$$

$$R \vee (S \wedge T) \equiv (R \vee S) \wedge (R \vee T)$$

$$S \vee \neg S \equiv \mathrm{True}$$

$$S \wedge \mathrm{True} \equiv S$$

Syntactic transformations: Examples

Example 4: a) Transform the statement $S_4 = (p \land q) \rightarrow (q \leftrightarrow r)$ into a semantically equivalent form where only the negation, disjunction, and conjunction operators are used; negations should only occur within literals.

b) Simplify the statement as far as possible. **c)** Transform $\neg S_4$ in the same way.

$$
\begin{aligned}
S_4 &\equiv \neg(p \land q) \lor (q \leftrightarrow r) \\
&\equiv \neg p \lor \neg q \lor (q \land r) \lor (\neg q \land \neg r) \\
&\equiv \neg p \lor ((\neg q \lor q) \land (\neg q \lor r)) \lor (\neg q \land \neg r) \\
&\equiv \neg p \lor (\text{True} \land (\neg q \lor r)) \lor (\neg q \land \neg r) \\
&\equiv \neg p \lor \neg q \lor r \lor (\neg q \land \neg r) \\
&\equiv \neg p \lor \neg q \lor ((r \lor \neg q) \land (r \lor \neg r)) \equiv \neg p \lor \neg q \lor ((r \lor \neg q) \land \text{True}) \\
&\equiv \neg p \lor \neg q \lor r \lor \neg q \\
&\equiv \neg p \lor \neg q \lor r
\end{aligned}
$$

$R \lor (S \land T) \equiv (R \lor S) \land (R \lor T)$

$S \lor \neg S \equiv \text{True}$

$S \land \text{True} \equiv S$

$S \lor S \equiv S$

Syntactic transformations: Examples

Example 4: a) Transform the statement $S_4 = (p \wedge q) \rightarrow (q \leftrightarrow r)$ into a semantically equivalent form where only the negation, disjunction, and conjunction operators are used; negations should only occur within literals.

b) Simplify the statement as far as possible. **c)** Transform $\neg S_4$ in the same way.

$$
\begin{aligned}
S_4 &\equiv \neg(p \wedge q) \vee (q \leftrightarrow r) \\
&\equiv \neg p \vee \neg q \vee (q \wedge r) \vee (\neg q \wedge \neg r) \\
&\equiv \neg p \vee ((\neg q \vee q) \wedge (\neg q \vee r)) \vee (\neg q \wedge \neg r) \\
&\equiv \neg p \vee (\text{True} \wedge (\neg q \vee r)) \vee (\neg q \wedge \neg r) \\
&\equiv \neg p \vee \neg q \vee r \vee (\neg q \wedge \neg r) \\
&\equiv \neg p \vee \neg q \vee ((r \vee \neg q) \wedge (r \vee \neg r)) \\
&\equiv \neg p \vee \neg q \vee r \vee \neg q \\
&\equiv \neg p \vee \neg q \vee r
\end{aligned}
$$

$$
\begin{aligned}
\neg S_4 &\equiv \neg(\neg p \vee \neg q \vee r) \\
&\equiv \neg\neg p \wedge \neg\neg q \wedge \neg r \\
&\equiv p \wedge q \wedge \neg r
\end{aligned}
$$

De Morgan's laws

$\neg(R \vee S) \equiv \neg R \wedge \neg S$

$\neg(R \wedge S) \equiv \neg R \vee \neg S$

Syntactic transformations: Examples

Example 5: Transform $S_{3c} = p_0 \rightarrow (p_1 \rightarrow (p_2 \rightarrow (p_3 \rightarrow (p_4 \rightarrow (p_5 \rightarrow (p_6 \rightarrow (p_7 \rightarrow p_0)))))))$ into a semantically equivalent form where only the negation, disjunction, and conjunction operators are used; negations should only occur within literals.

$$
\begin{aligned}
S_{3c} &\equiv p_0 \rightarrow (p_1 \rightarrow (p_2 \rightarrow (p_3 \rightarrow (p_4 \rightarrow (p_5 \rightarrow (p_6 \rightarrow (p_7 \rightarrow p_0))))))) \\
&\equiv \neg p_0 \lor (p_1 \rightarrow (p_2 \rightarrow (p_3 \rightarrow (p_4 \rightarrow (p_5 \rightarrow (p_6 \rightarrow (p_7 \rightarrow p_0))))))) \\
&\equiv \neg p_0 \lor \neg p_1 \lor (p_2 \rightarrow (p_3 \rightarrow (p_4 \rightarrow (p_5 \rightarrow (p_6 \rightarrow (p_7 \rightarrow p_0)))))) \\
&\equiv \ldots \\
&\equiv \neg p_0 \lor \neg p_1 \lor \neg p_2 \lor \neg p_3 \lor \neg p_4 \lor \neg p_5 \lor \neg p_6 \lor \neg p_7 \lor p_0 \\
&\equiv (\neg p_0 \lor p_0) \lor \neg p_1 \lor \neg p_2 \lor \neg p_3 \lor \neg p_4 \lor \neg p_5 \lor \neg p_6 \lor \neg p_7 \\
&\equiv \text{True} \lor (\neg p_1 \lor \neg p_2 \lor \neg p_3 \lor \neg p_4 \lor \neg p_5 \lor \neg p_6 \lor \neg p_7) \\
&\equiv \text{True}
\end{aligned}
$$

Entailment and normal forms: Repetition

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Entailment: Summary

The statement R **entails** the statement S if *all valuations that fulfill R also fulfill S*. In other words: If R is True, we know that S is True as well.

A **valuation** for which a statement S becomes True is also called a **model** of S.

Semantic equivalence	**Entailment**
$R \equiv S$	$R \vDash S$
("The statements R and S are semantically equivalent")	("The premise R entails the conclusion S")
***R* and *S* have the same models:** Their truth tables are the same.	**All models of *R* are also models of *S*.** S may still be True where R is False, *i.e.*, S may have more models than R.
The statement $R \leftrightarrow S$ is a tautology.	The statement $R \rightarrow S$ is a tautology.

Entailment: Discussion

Statement R entails statement S,

$$\boxed{R \vDash S,}$$

Every model of R
is also a model of S.

which is semantically equivalent to statement T,

$$\boxed{S \equiv T.}$$

S and T have
the same models.

Provide a justified answer to the following question:

Do we know enough to conclude that $R \vDash T$?

Entailment: Discussion

Statement *R* entails statement *S*,

$$\boxed{R \models S,}$$

which is a contradiction,

$$\boxed{S \equiv \text{False}.}$$

We could also just have written: $R \models$ False.
That is, *R* entails the contradiction "False."

What do we know about the truth table of *R*?

Normal forms: Summary

Valuation: Assignment of truth values to all the atomic statements. For n atomic statements, there are 2^n possible valuations. Each row of a truth table corresponds to one valuation.

Literals: Atomic statements p, q, …, and their negations $\neg p$, $\neg q$, …

Clauses: A conjunction ("and") of literals, such as $p \wedge \neg q \wedge \neg r$, is a conjunctive clause. A disjunction ("or") of literals, such as $\neg p \vee q \vee r$, is a disjunctive clause.

DNF:

- A statement is in DNF if it is a **disjunction of conjunctive clauses**.
- It is in **full DNF** if all atomic statements appear in all conjunctive clauses.
- The full DNF version of a truth table has one clause per **True** valuation.

CNF:

- A statement is in CNF if it is a **conjunction of disjunctive clauses**.
- It is in **full CNF** if all atomic statements appear in all disjunctive clauses.

Normal forms: Summary

CNF:

- A statement is in CNF if it is a **conjunction of disjunctive clauses**.
- It is in **full CNF** if all atomic statements appear in all disjunctive clauses.
- The full CNF version of a truth table has one clause per **False** valuation.
- For a **tautology**, no False valuations exist; the full CNF is just "True."

- Tautology in full CNF: True
- Contradiction in full CNF: $(\neg p \lor \neg q) \land (\neg p \lor q) \land (p \lor \neg q) \land (p \lor q)$

DNF:

- A statement is in DNF if it is a **disjunction of conjunctive clauses**.
- It is in **full DNF** if all atomic statements appear in all conjunctive clauses.
- The full DNF version of a truth table has one clause per **True** valuation.
- For a **contradiction**, no True valuations exist; the full DNF is just "False."

- Tautology in full DNF: $(\neg p \land \neg q) \lor (\neg p \land q) \lor (p \land \neg q) \lor (p \land q)$
- Contradiction in full DNF: False

Normal forms: Example

Example 6: Transform $S_6 = (\neg p \lor \neg q \lor r) \land (p \lor \neg q \lor \neg r) \land (\neg p \lor \neg q \lor \neg r)$ from full CNF to full DNF.

p	q	r	$(\neg p \lor \neg q \lor r)$	$(p \lor \neg q \lor \neg r)$	$(\neg p \lor \neg q \lor \neg r)$	S_6
False	False	False	True	True	True	True
False	False	True	True	True	True	True
False	True	False	True	True	True	True
False	True	True	True	**False**	True	False
True	False	False	True	True	True	True
True	False	True	True	True	True	True
True	True	False	**False**	True	True	False
True	True	True	True	True	**False**	False

Normal forms: Example

Example 6: Transform $S_6 = (\neg p \lor \neg q \lor r) \land (p \lor \neg q \lor \neg r) \land (\neg p \lor \neg q \lor \neg r)$ from full CNF to full DNF.

p	q	r	$(\neg p \lor \neg q \lor r)$	$(p \lor \neg q \lor \neg r)$	$(\neg p \lor \neg q \lor \neg r)$	S_6
False	False	False	True	True	True	**True**
False	False	True	True	True	True	**True**
False	True	False	True	True	True	**True**
False	True	True	True	False	True	False
True	False	False	True	True	True	**True**
True	False	True	True	True	True	**True**
True	True	False	False	True	True	False
True	True	True	True	True	False	False

$$S_6 \equiv (\neg p \land \neg q \land \neg r) \lor (\neg p \land \neg q \land r) \lor (\neg p \land q \land \neg r) \lor (p \land \neg q \land \neg r) \lor (p \land \neg q \land r)$$

Entailment and inference: Summary

Entailment

$R_0, R_1, \ldots \vDash S$

("The premises R_0, R_1, … entail the conclusion S")

All models of all premises are also models of S.

Where R_0, R_1 etc. are all True, S must also be True.

But S can have more True entries than $R_0 \land R_1 \land \ldots$

The statement $(R_0 \land R_1 \land \ldots) \rightarrow S$ is a tautology.

We can determine entailment *semantically*, from *truth tables*.

We can also determine it **syntactically**, using **inference rules**.

$R \rightarrow S,\ R \vDash S$

$S \rightarrow R,\ \neg R \vDash \neg S$

$R \vDash \text{True}$

$\text{False} \vDash S$

$R,\ S \vDash R \land S$

$R \lor S,\ \neg R \lor T \vDash S \lor T$

Inference technique for CNF: Resolution

For premises expressed in **conjunctive normal form (CNF)**, a simple and powerful rule is available for deducing new disjunctive clauses: **Resolution**.

General algorithm:

- Start with given set of disjunctive clauses.
- Iteratively, as long as possible,
  - find clauses with opposite literals;
  - add the inferred clause (if new);
  - if nothing new can be deduced, terminate.

inference rule for resolution

$$\boxed{R \vee S,\ \neg R \vee T \ \vDash\ S \vee T}$$

Applied to literals L_0, L_1, ..., M_0, M_1, … the inference rule for resolution becomes:

$$(p \vee L_0 \vee L_1 \vee \ldots) \wedge (\neg p \vee M_0 \vee M_1 \vee \ldots) \wedge \ldots \ \vDash\ (L_0 \vee L_1 \vee \ldots \vee M_0 \vee M_1 \vee \ldots)$$

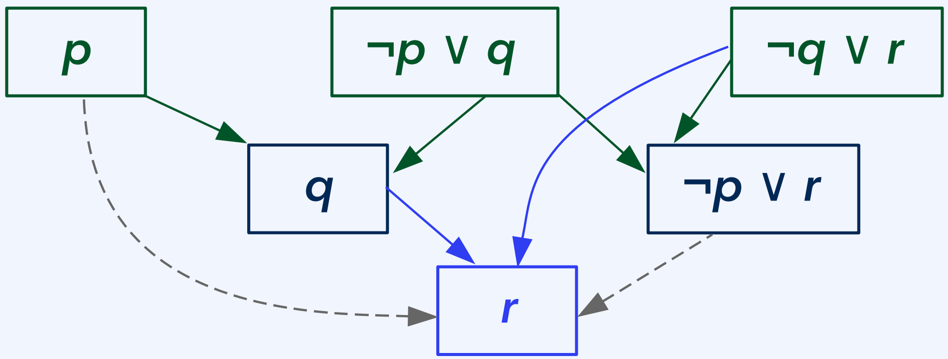
Resolution is a **complete** calculus for **(un)satisfiability**. It finds contradictions.

Inference by resolution: Example

in CNF, $\neg p \lor q$ in CNF, $\neg q \lor r$ in CNF, p

Let us begin with the premises $(p \rightarrow q)$, $(q \rightarrow r)$, and p.

These premises entail r. How would this consequence be found by resolution?



Applied to literals L_0, L_1, ..., M_0, M_1, … the inference rule for resolution becomes:

$$(p \lor L_0 \lor L_1 \lor \ldots) \land (\neg p \lor M_0 \lor M_1 \lor \ldots) \land \ldots \models (L_0 \lor L_1 \lor \ldots \lor M_0 \lor M_1 \lor \ldots)$$

Resolution is a **complete** calculus for **(un)satisfiability**. It finds contradictions.

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