Number of truth tables: Tutorial 4.3 problem

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Satisfiability of propositional logic statements

Question 4.3.1: Are the following propositional logic statements satisfiable?

$S_a \quad = \quad ((p \wedge r) \rightarrow (q \vee r)) \leftrightarrow (s \wedge \neg s)$

False

$S_b \quad = \quad (p \vee \neg q \vee \neg r \vee \neg s) \wedge ((p \wedge r) \rightarrow (q \vee r)) \wedge \neg p \wedge q \wedge r \wedge s$

$\boldsymbol{p \vee \neg q \vee \neg r \vee \neg s \equiv \neg(\neg p \wedge q \wedge r \wedge s)}$

$S_c \quad = \quad (p \leftrightarrow \neg p) \rightarrow ((p \leftrightarrow \neg q) \wedge (q \leftrightarrow \neg r) \wedge (r \leftrightarrow \neg s) \wedge (s \leftrightarrow \neg t) \wedge (t \leftrightarrow \neg p))$

False

$S_d \quad = \quad \neg S_c$

Satisfiability of propositional logic statements

Question 4.3.1: Are the following propositional logic statements satisfiable?

$S_a \quad = \quad ((p \land r) \rightarrow (q \lor r)) \leftrightarrow (s \land \neg s)$

True **False**

If $p \land r$ is True, r is True, and hence $q \lor r$ is also True.

$(p \land r) \rightarrow (q \lor r)$ is a tautology

$(s \land \neg s)$ is a contradiction

S_a is a contradiction

$S_b \quad = \quad (p \lor \neg q \lor \neg r \lor \neg s) \land ((p \land r) \rightarrow (q \lor r)) \land \neg p \land q \land r \land s$

There is no valuation for which $(p \lor \neg q \lor \neg r \lor \neg s)$ and $(\neg p \land q \land r \land s)$ are both True. Therefore, **S_b is a contradiction**.

$S_c \quad = \quad (p \leftrightarrow \neg p) \rightarrow ((p \leftrightarrow \neg q) \land (q \leftrightarrow \neg r) \land (r \leftrightarrow \neg s) \land (s \leftrightarrow \neg t) \land (t \leftrightarrow \neg p))$

False

An implication can only become False if the left-hand side is True. That is impossible here. **S_c is a tautology** (and, hence, **satisfiable**).

$S_d \quad = \quad \neg S_c$

Since S_c is a tautology, **$\neg S_c$ is a contradiction**.

Enumeration of truth tables

Question 4.3.2a: Are there ≥1000 truth tables with four atomic statements?

Let us consider the case where there are $n = 2$ atomic statements, p and q:

p	q	T_0	T_1	T_2	T_3	…	T_{14}	T_{15}
False	False	False	False	False	False	…	True	True
False	True	False	False	False	False	…	True	True
True	False	False	False	True	True	…	True	True
True	True	False	True	False	True	…	False	True

For n atomic statements, there are 2^n different valuations; therefore, there are 2^n entries – each of which may be True or False – in the truth tables.

The total number of different truth tables is therefore $2^{(2^n)}$.

$n = 2$: there are $2^4 = 16$ truth tables; $n = 4$: there are $2^{16} = 65{,}536$ truth tables.

Enumeration of truth tables

Question 4.3.2a: Are there ≥1000 truth tables with four atomic statements? Yes!

p	q	r	s	T_0	T_1	T_2	…	T_{65535}
False	False	False	False	False	False	False	…	True
False	False	False	True	False	False	False	…	True
False	False	True	False	False	False	False	…	True
False	False	True	True	False	False	False	…	True
False	True	False	False	False	False	False	…	True
False	True	False	True	False	False	False	…	True
False	True	True	False	False	False	False	…	True
…	…	…	…	…	…	…	…	…
True	True	False	True	False	False	False	…	True
True	True	True	False	False	False	True	…	True
True	True	True	True	False	True	False	…	True

In general, there are $2^{(2^n)}$ truth tables; $n = 4$: there are $2^{16} = 65{,}536$ truth tables.

Enumeration of truth tables

Task 4.3.2b: Find two (semantically) different statements both entailed by *R*:

There, the premise R was given by $\neg((p \rightarrow q) \rightarrow (\neg q \wedge \neg r))$.

<u>**Entailment**</u>

$$R \vDash S$$

("The premise *R* entails the conclusion *S*")

All models of *R* are also models of *S*.

S may still be True where *R* is False, *i.e.*,
S may have more models than *R*.

The statement $R \rightarrow S$ is a tautology.

Two statements entailed by *R*:

$$R \vDash R$$

$$R \vDash \text{True}$$

Enumeration of truth tables

Question 4.3.2c: How many propositional logic statements S, using p, q, and r only, with different truth tables exist, such that $R \vDash S$?

There, the premise R was given by $\neg((p \rightarrow q) \rightarrow (\neg q \land \neg r))$.

p	q	r	$(p \rightarrow q)$	$(\neg q \land \neg r)$	$((p \rightarrow q) \rightarrow (\neg q \land \neg r))$	R
False	**False**	**False**	True	True	True	False
False	False	True	**True**	**False**	False	True
False	True	False	**True**	**False**	False	True
False	True	True	**True**	**False**	False	True
True	**False**	**False**	False	True	True	False
True	**False**	True	False	False	True	False
True	True	False	**True**	**False**	False	True
True	True	True	**True**	**False**	False	True

All the True entries in the truth table of R must also be True for S.

Enumeration of truth tables

Question 4.3.2c: How many propositional logic statements S, using p, q, and r only, with different truth tables exist, such that $R \vDash S$?

There, the premise R was given by $\neg((p \rightarrow q) \rightarrow (\neg q \wedge \neg r))$.

p	q	r	$(p \rightarrow q)$	$(\neg q \wedge \neg r)$	$((p \rightarrow q) \rightarrow (\neg q \wedge \neg r))$	R
False	**False**	**False**	True	True	**True**	**False**
False	False	True	**True**	**False**	False	True
False	True	False	**True**	**False**	False	True
False	True	True	**True**	**False**	False	True
True	**False**	**False**	False	True	**True**	**False**
True	**False**	True	False	False	**True**	**False**
True	True	False	**True**	**False**	False	True
True	True	True	**True**	**False**	False	True

All the True entries in the truth table of R must also be True for S.
There are **3 False entries** in the truth table of R; they may be True or False for S.

Therefore, there are $2^3 = 8$ semantically different $S_0, \ldots, S_7$ entailed by R.